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Note

A Method of Spectral Analysis Applied to Periodic and Pseudoperiodic Signals

INTRODUCTION

The analysis of a large number of physical phenomena requires calculations of periodic and pseudoperiodic components of various signals given by the output of measurement devices. A set of discrete values corresponding to the time evolution of a signal must generally be analyzed. These values are usually obtained by sampling at regular time intervals of parameters describing the evolution of a system. Numerical results must often be analyzed in a similar fashion, as is shown for instance in [1, 2, 3].

The object of this note is to present a method based on the classical technique of fast Fourier transform for identifying the pseudoperiodic components of a signal or a function defined by a set of discrete values.

The method is illustrated by results for both cases when the signal contains noise or not.

THE METHOD

Let us consider a signal $g(t)$ sampled at regular time $t_r = rT/N$, where T is the duration of sampling in seconds, N is the number of samples, and we have $0 \leqslant r \leqslant N-1$, in other words, T is the width of the rectangular window function which will be used throughout the analysis of the complete signal $g(t)$ defined on the time interval $-\infty < t < +\infty$.

Furthermore we define

$$g_r = g(t_r) \qquad r \in [0, N-1]. \tag{1}$$

These sampled data are supposed to satisfy the Shannon condition (sampling frequencies are greater than twice the largest frequency of the signal), and it is assumed that no aliasing phenomenon is present [4, 5]. Numerically, it is simple and very fast by using an FFT algorithm to compute the discrete Fourier transform of the function $g(t)$ corresponding to the frequency interval $2\pi/T$; we thus have

$$G_j = (1/N) \sum_{z=0}^{N-1} g_r \exp(-i2\pi rj/N) \qquad j \in [0, N/2]. \tag{2}$$

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In general, the values G_j do not provide the exact spectrum of the function $g(t)$; in particular; if periodic components are present in the signal, the peaks of the spectrum are not equal to amplitude of the periodic components, except for very special cases [6, 7] ("leakage effect"). Thus it is not possible to determine the complex amplitude and frequency of the various components in the signal from the knowledge of the values G_j. The present method is a lineshape fitting method that makes it possible to obtain the exact values of amplitude and frequency from the values G_j in all cases.

The method requires that the signal consists of a superposition of a finite number of well-distinct and weakly-damped periodic components; we thus have

$$g_r = (\tfrac{1}{2}) \sum_{k=0}^{N-1} A_k \exp(i\Omega_k t_r) + \bar{A}_k \exp(-i\bar{\Omega}_k t_r) + B(t_r) \tag{3a}$$

for a real signal $g(t)$ or

$$g_r = \sum_{k=0}^{N-1} A_k \exp(i\Omega_k t_r) + B(t_r) \tag{3b}$$

for a complex signal, where we have

$$A_k = a_k + ib_k \tag{4}$$

$$\Omega_k = i\lambda_k + \omega_k \tag{5}$$

with a_k, b_k, λ_k, and ω_k real and $|\lambda_k| \ll 1$, and $B(t_r)$ represents a low-level noise.

In Eqs. (3a) and (3b), the integer k takes values between 0 and $N-1$; however, only some values of A_k are nonzero since the signal is assumed to consist of distinct components. In this study, the analysis will only be presented for the case of a complex signal. For a real signal, the analysis needs to be slightly modified but the procedure for numerical calculations is essentially identical; specifically, the amplitude of a real signal as computed by using an algorithm based on the input of a complex signal, only needs to be multiplied by two.

Equations (3b) and (2) yield

$$G_j = \left(\frac{1}{N}\right) \sum_{k=0}^{N-1} A_k \frac{1-\exp[-\lambda_k T + i(\omega_k T - 2\pi j)]}{1-\exp[-\lambda_k T + i(\omega_k T - 2\pi j)]/N} + \left(\frac{1}{N}\right) \sum_{r=0}^{N-1} B_r \exp\left(\frac{-i2\pi rj}{N}\right). \tag{6}$$

Equation (6) shows that there is no obvious direct relation between the values G_j and A_k except for those components for which the frequencies ω_k are integer multiples of the frequency interval $\Delta\omega = 2\pi/T$. For these particular components with $j = K$ and $\omega_K = K2\pi/T$, Eq. (7a) can be obtained from Eq. (6)

$$G_K = A_K \frac{1-\exp(-\lambda_K T)}{\lambda_K T} + \left(\frac{1}{N}\right) \sum_{k=0, k\neq K}^{N-1} A_K \frac{1-\exp(-\lambda_k T)}{1-\exp(-\lambda_k T + i2\pi(k-K))/N} + \left(\frac{1}{N}\right) \sum_{r=0}^{N-1} \exp(-i2\pi rK/N)\, B_r \tag{7a}$$

provided that λ_K is small. It can be seen that if all the values λ_k vanish for k in $[0, N-1]$ we have

$$G_k = A_K + C(K). \tag{7b}$$

In the latter case, G_K thus immediately yields the value of A_K, disregarding the contribution $C(K)$ of the noise. The foregoing property of the discrete Fourier transform is utilized in methods where T can be matched to the valuess of ω_K. One of these methods, so-called "tracking method," automatically performs the adjustment of T by means of electronical apparatus [6].

More generally, the frequency of a component can be related to two successive integer multiples of $\Delta\omega$, as follows:

$$K\Delta\omega \leqslant \omega_K \leqslant (K+1)\,\Delta\omega \tag{8a}$$

$$(K-1)\,\Delta\omega \leqslant \omega_K \leqslant K\Delta\omega. \tag{8b}$$

In practice, K corresponds to the value of j for which G_j is a peak in the spectrum, even if the leakage effect is present. Equations (8) can be expressed as follows:

$$\omega_K = (K + \varepsilon_K)\, 2\pi/T \qquad \text{with} \quad -1 \leqslant \varepsilon_K \leqslant 1. \tag{8c}$$

The present method consists in determining ε_K, from which ω_K is then obtained by using Eq. (8c). In the vicinity of a peak corresponding to $j = K$, Eq. (6) yields

$$G_K = A_K \frac{1-\exp(-\lambda_K T + i2\pi\varepsilon_K)}{\lambda_K T - i2\pi\varepsilon_K} + \left(\frac{1}{N}\right) \times \sum_{\substack{k=0\\ k\neq k_0}}^{N-1} A_K \frac{1-\exp(-\lambda_k T + i2\pi\ \varepsilon_K)}{1-\exp((-\lambda_k T + i2\pi(\varepsilon_k + k - K))/N} + C(K). \tag{9}$$

If the terms of order $1/N$ are neglected, we have

$$G_K = A_K \frac{1-\exp(-\lambda_K T + i2\pi\varepsilon_K)}{\lambda_K T - i2\pi\varepsilon_K} + C(K). \tag{9a}$$

Similarly for $j=K+1$ and $j=K-1$, we have

$$G_{K+1}=A_K\frac{1-\exp(-\lambda_K T+i2\pi\varepsilon_K)}{\lambda_K T-i2\pi\varepsilon_K-i2\pi}+C(K+1) \tag{9b}$$

$$G_{K-1}=A_K\frac{1-\exp(-\lambda_K T+i2\pi\varepsilon_K)}{\lambda_K T-2i\pi\varepsilon_K+i2\pi}+C(K-1). \tag{9c}$$

Let us assume that the noise is a filtered white noise and that its level is low. The Fourier transform $C(K)$ of the noise $B(t)$ is then gradually varying between the values $K-1$, K, $K+1$, so that the contribution of the noise can be neglected in the differences G_K-G_{K+1} and G_K-G_{K-1}. Let A represent the value of the ratio $(G_K-G_{K+1})/(G_K-G_{K-1})$, in which the contribution of the noise is ignored as was just explained.

Equations (9a), (9b), (9c) then yield

$$A=-(Z_K+i2\pi)/(Z_K-i2\pi), \tag{10}$$

where Z_K is defined as

$$Z_K=-\lambda_K T+i2\pi\varepsilon_K. \tag{11}$$

From Eq. (10) we may readily obtain

$$Z_K=i2\pi(A-1)/(A+1).$$

By identifying the real and imaginary parts of Z_K in Eq. (11), we then can determine the values of λ_K and ε_K. The complex frequency Ω_K defined in Eq. (5) then becomes

$$\Omega_K=i\lambda_K+(K+\varepsilon_K)\,2\pi/T, \tag{12}$$

where Eq. (8c) was used.

The contribution of the noise can be eliminated in a similar fashion by considering the sum $2G_K-G_{K+1}-G_{K-1}$. Equations (9a), (9b), (9c) then yield the amplitude A_K:

$$A_K=(2G_K-G_{K+1}+G_{K-1})\,\frac{Z_K(Z_K^2+4\pi^2}{8\pi^2(1-\exp(-Z_K))}. \tag{13}$$

In summary, the method consists in computing the FFT of the signal for $N=2^\alpha$, in detecting the peaks $j=K$ in the spectrum and in calculating ε_K and λ_K from Eq. (11), the complex frequency Ω_K from Eq. (12), and the complex amplitude A_K from Eq. (13).

Comparison of the Present Method with that of Reference [8]

The method presented in the present study is similar to that used by Feit and Fleck [8]. The function $g(t)$ is multiplied by a Hanning window function in [8]. The number ε_K, referred to as the "offset parameter" by the authors of [8], is given by a second-degree algebraic equation instead of a first-degree equation in the present study. As far as can be seen, the authors of [8] have not applied their method to noised signals. Equation (10) in the present study shows how the noise contribution can be eliminated in part in the calculation of Ω_K and A_K. These two points (the simple expression of the frequency shift and the elimination in part of the noise contribution) appear as the advantages of the rectangular window in the present work.

Numerical results obtained by using the two methods are compared in Table I for a signal defined by

$$g(t) = \sum_{j=1}^{10} \exp(i\beta_j t)$$

where $\beta_j = \beta_j^r + i\beta_j^i$ and $A_j = 1$.

The values of frequency, damping, and amplitude predicted by the two methods can be seen in very close agreement. It should be noticed that a global method as shown in [8] is more accurate for a signal without noise. This method consists of first making the previously described local estimation which is supposed to yield

TABLE I

Comparison between the Actual Values of the Complex Frequencies for a Spectrum with Ten Peaks and the Values Predicted by the Method Used in [8] and in the Present Study

Actual			Computed (Ref. [8]) (a)			Computed (present method) (b)			
A_n	β_n^r	β_n^i	$\lvert A_n\rvert$	β_n^r	$\beta_n^i \times 10^8$	$\lvert A_n\rvert$	$\arg(A_n)$ deg $^\circ \times 10^3$	β_n^r	$\beta_n^{i\,8}$ $\times 10^8$
1.	800.	0.	0.9999992	800.0000	6.91	0.9999942	0.68	800.0000	10.5
1.	760.	0.	0.9999981	760.0000	−11.4	0.9999971	−1.03	760.0000	1.43
1.	720.	0.	0.9999984	720.0000	67.9	0.9999971	−0.93	720.0000	3.62
1.	680.	0.	0.9999994	680.0000	−15.7	0.9999964	−0.70	680.0000	6.99
1.	640.	0.	0.9999996	640.0000	−164.	0.9999962	−0.61	640.0000	9.21
1.	600.	0.	0.9999985	600.0000	−70.9	0.9999964	0.41	600.0000	10.4
1.	560.	0.	0.9999980	560.0000	−10.3	0.9999963	0.22	560.0000	10.6
1.	520.	0.	0.9999980	520.0000	6.94	0.9999967	0.02	520.0000	9.68
1.	480.	0.	0.9999983	480.0000	−16.9	0.9999977	0.16	480.0000	7.25
1.	440.	0.	1.0000020	440.0000	−33.4	0.9999992	0.10	440.0000	2.02

Note. The amplitudes A_n are complex in the present method). $g(t) = \sum_{j=1}^{10} \exp(i\beta_j t)$; $\beta_j = \beta_j^r + i\beta_j^i$; $A_j = 1$.

accurate frequencies. Then, holding the frequencies fixed, a global least square fit of complex amplitudes is made by using an over determinate linear system (solved by a Householder method). The results corresponding to this global method are not reported in this paper.

Numerical Results

Tables II and III present numerical results corresponding to the signal

$$g(t) = \sum_{j=1}^{10} A_j \exp(-i\beta_j t) + B(t).$$

In Table II, no noise is present, but two components having widely different amplitudes (1. and 0.005) and two other components with frequencies that are very close (600. and 603.) are considered. The accuracy of the predicted frequencies can be seen to be very good; however, the accuracy of the calculated amplitude and damping is noticeably reduced by the proximity of two frequencies. Results for a noised signal are presented in Table III. It can be seen that the frequencies are predicted with high accuracy, but that the calculated amplitudes exhibit errors of 3×10^{-3} to 4×10^{-3} instead of 4×10^{-6} to 5×10^{-6} in the case of signal without noise. The noise was generated by using the well-known subroutines RANDU and GAUSS for a set of values of data with a time interval of 0.00375 s.

Calculations were performed on a NAS9050. Other calculations not reported

TABLE II

Analysis of a Signal Having Two Widely Different Amplitudes and Two Very Close Frequencies

Actual values				Computed values			
n	A_n	β_n^r	β_n^i	$\lvert A_n \rvert$	$\arg(A_n)$ deg. $\times 10^3$	β_n^r	$\beta_n^i \times 10^8$
1	1.	800.	0	0.9999922	−0.385	800.0000	−23.0
2	1.	760.	0	0.9999950	−0.675	760.0000	−10.1
3	1.	720.	0	1.0000001	−0.602	720.0000	7.59
4	0.005	680.	0	0.4994×10^{-2}	75.0	650.0000	2210
5	1.	640.	0	0.9999918	0.118	640.0000	−23.6
6	1.	603.	0	1.0002924	2.70	603.0000	1050
7	1.	600.	0	0.9989991	−6.08	600.0000	−3200
8	1.	520.	0	0.9999928	−0.398	520.0000	−21.6
9	1.	480.	0	0.9999958	−0.615	480.0000	−8.44
10	1.	440.	0	1.000001	−0.475	440.0000	10.1

Note. The signal is not noised.

TABLE III

Analysis of a Ten-Peak Spectrum for Signal with Noise Characterized by a Null Average and a Standard Deviation $\sigma = 0.1$

	Actual values			Computed values			
n	A_n	β_n^r	β_n^i	$\lvert A_n \rvert$	$\arg(A_n)$ deg.	β_n^r	$\beta_n^i \times 10^4$
1	1.	800.	0	0.992910	−0.036	800.0000	−1.10
2	1.	760.	0	0.996323	0.149	760.0000	−1.35
3	1.	720.	0	0.998309	−0.175	720.0000	0.10
4	1.	680.	0	1.00414	−0.336	680.0001	2.21
5	1.	640.	0	0.996905	0.208	640.0000	−1.32
6	1.	600.	0	1.001869	−0.090	600.0000	0.40
7	1.	560.	0	0.997127	−0.222	560.0000	−0.80
8	1.	520.	0	0.993258	0.053	520.0000	−0.17
9	1.	480.	0	1.004288	−0.015	480.0000	1.14
10	1.	440.	0	0.996067	−0.301	440.0002	−0.50

here have been performed for signals with frequencies up to 1000 Hz and good results were obtained.

An alternative method for determining the pseudoperiodic components of a signal is Prony's method [2, 11] or its generalization [12, 13].

Conclusion

The method presented in the present study is useful for determining the pseudoperiodic components of a signal defined by a set of discrete values. The frequencies can be predicted with high accuracy whether or not the signal is noised. The accuracy of the calculated damping and to a lesser extent of the amplitude is reduced, but nonetheless remains quite reasonable, when the signal is noised or has frequencies very close to one another. The method has been applied successfully to analyze pseudo-periodic phenomena in hydrodynamics for which the frequencies usually are lower than 15 Hz [9, 10].

References

1. M. D. Feit, J. A. Fleck Jr., and A. Steiger, *J. Comput. Phys.* **47** (1982), 412.
2. A. E. Siegman and H. Y. Miller, *Appl. Opt.* **9**, No. 12 (1970).
3. Y. Lecointe and J. Piquet, *in* "Numerical Methods and Methodology in Heat Transfer: Proceedings 2nd National Symposium, Hemisphere, Washington, D.C. 1982.
4. O. E. Brigham, "The Fast Fourier Transform," Prentice–Hall, Englewood Cliffs, N.J., 1974.
5. S. A. Orszag, *Stud. Appl. Math.* **51**, 3 (1972).
6. K. Somla, *in* "Numerical Analysis on Dynamic Ship Structure: Euromech Colloquium 112 (1972)."
7. F. J. Harris, *Proc. IEEE* **66** (1978), 5.
8. M. D. Feit and J. A. Fleck, Jr., *Appl. Opt.* **20**, No. 5 (1981).

9. P. Sulmont and D. Rajaona, "Analogic Signal Analysis Method by Mean of Fast Fourier Transform," ATMA **82** (1982), 483, Paris.
10. D. Rajaona and P. Sulmont, "Attenuation of Vortex-Shedding in a Flow Past a Circular Cylinder," ATMA **83** (1983), 217, Paris.
11. D. Kammler, *J. Math. Anal. Appl.* **57** (1977), 960.
12. S. M. Kay and S. L. Marple, Jr., *Proc. IEEE* **69** (1981), 1380.
13. J. C. Daubisse, "Sur quelques méthodes numériques spécifiques de l'Hydrodynamique," Thesis, Université de Nantes, 1984.

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